Image Geometry Through Multiscale Statistics

by

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Abstract

This study in the statistics of scale space begins with an analysis of noise propagation of multiscale differential operators for image analysis. It also presents methods for computing multiscale central moments that characterize the probability distribution of local intensities. Directional operators for sampling oriented local central moments are also computed and principal statistical directions extracted, reflecting local image geometry. These multiscale statistical models are generalized for use with multivalued data.

The absolute error in normalized multiscale differential invariants due to spatially uncorrelated noise is shown to vary non-monotonically across order of differentiation. Instead the absolute error decreases between zeroth and first order measurements and increases thereafter with increasing order of differentiation, remaining less than the initial error until the third or fourth order derivatives are taken.

Statistical invariants given by isotropic and directional sampling operators of varying scale are used to generate local central moments of intensity that capture information about the local probability distribution of intensities at a pixel location under an assumption of piecewise ergodicity. Through canonical analysis of a matrix of second moments, directional sampling provides principal statistical directions that reflect local image geometry, and this allows the removal of biases introduced by image structure. Multiscale image statistics can thus be made invariant to spatial rotation and translation as well as linear functions of intensity.

These new methods provide a principled means for processing multivalued images based on normalization by local covariances. They also provide a basis for choosing control parameters in variable conductance diffusion.
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# Contents

Abstract ........................................................................................................................................... i
Acknowledgements ........................................................................................................................... ii
Contents .............................................................................................................................................. iv
List of Tables ..................................................................................................................................... vii
List of Figures ................................................................................................................................... viii
List of Symbols ................................................................................................................................. xi

## Chapter 1 Introduction ................................ ................................ .................................................. 1
  1.1. A multiscale approach to computer vision ................................................................. 2
  1.2. An integrated approach to early vision ................................................................. 3
  1.3. Driving issues .......................................................................................................... 4
  1.4. Thesis ....................................................................................................................... 5
  1.5. Overview ................................................................................................................ 6
  1.6. Contributions ......................................................................................................... 7

## Chapter 2 Background ................................ ................................ .................................................. 9
  2.1. Notation .................................................................................................................. 9
  2.2. Images .................................................................................................................. 9
    2.2.1. Images as a 2D manifold in n-space .......................................................... 10
    2.2.2. Digital Images ............................................................................................. 11
  2.3. Invariance .......................................................................................................... 12
    2.3.1. Gauge Coordinates .................................................................................... 13
  2.4. Scale Space ....................................................................................................... 14
    2.4.1. Differentiation ........................................................................................... 15
    2.4.2. The Gaussian as a unique Regular Tempered Distribution .................... 16
    2.4.3. Zoom Invariance ....................................................................................... 18
    2.4.4. Gaussian Scale Space ................................................................................ 18
  2.5. Image Statistics .................................................................................................... 19
    2.5.1. The Normal Density vs. the Gaussian Filter Kernel ............................... 19
    2.5.2. Noisy Images ............................................................................................. 20
    2.5.3. Statistical Measures as Invariants: Mahalanobis Distances ................. 20
    2.5.4. Calculating Central Moments ................................................................... 21
    2.5.5. Characteristic Functions .......................................................................... 23
      A Simple Univariate Example (a Gaussian Normal Distribution) .......... 23
      Bivariate Characteristic Functions ................................................................. 25
  2.6. Moment Invariants of Image Functions ..................................................................... 26

## Chapter 3 Normalized Scale Space Derivatives: ................................................................. 27
  3.1. Introduction and Background .................................................................................... 27
    3.1.1. Scale Space Differential Invariants ......................................................... 28
    3.1.2. Reconstruction of Sampled Images via the Taylor Expansion ............ 29
    3.1.3. Exploring the Properties of Scale-space Derivatives ......................... 30
List of Tables

Table 3.1. Variances of unnormalized scale space derivatives (order 0-6) of noisy 1D images (variance of input noise = $v_0$) ................................................................. 35
Table 3.2. Variances of unnormalized scale space derivatives of noisy 2D images (variance of input noise = $v_0$) for partial spatial derivatives to the fourth order (Adapted from Blom 1992) ................................................................. 35
Table 3.3. Variances of normalized scale space derivatives (order 0-6) of noisy 1D images (variance of input noise = $v_0$) ................................................................. 37
Table 3.4. Variances of normalized scale space derivatives of noisy 2D images for partial spatial derivatives to the fourth order (variance of input noise = $v_0$) ...................... 37
Table 3.5. Variances of both unnormalized and normalized scale space derivatives (order 0-6) of noisy 2D images (variance of input noise = $v_0$) .............................. 41
List of Figures

Figure 1.1. A segmentation example. (a) the original digital radiograph, (b) an image mask denoting segments, and (c) the classified segment mask, showing the hierarchical semantic organization of the skeletal system. ...........................................3

Figure 2.1. Three representations of an image. From left to right: (a) greyscale representation, (b) intensity surface, and (c) isophotes.......................................................10

Figure 2.2. The image in Figure 2.1 represented as a digital image with a raster resolution of 64 × 64 pixels. ........................................................................................................11

Figure 2.3. 2-D Gaussian derivative filter kernels through the 4th order. ................................................16

Figure 2.4. Top: Characteristic function for a zero mean Gaussian. Maclaurin approximating polynomials (a) n=2, (b) n=8, (c) n=10, and (d) n=16..............................................25

Figure 3.1. Propagated error of unnormalized 1D scale space derivatives (order 0-6). Each curve represents the ratio of variance of output to input noise of the linear unnormalized derivative of Gaussian filter vs. scale $\sigma$. Plot is on a log-log scale. 38

Figure 3.2. Plot of the propagated error of normalized 1D scale space derivatives (order 0-6). Each curve represents the ratio of variance of output to input noise of the linear normalized derivative of Gaussian filter vs. scale $\sigma$. Plot is on a log-log scale. 38

Figure 3.3. Plot of the propagated error of normalized 1D scale-space derivatives (order 0-6). Curve represents the ratio of variance of output to input noise of the linear unnormalized derivative of Gaussian filter vs. order of differentiation. Plot is on a log scale. ............................................................39

Figure 3.4. Propagated error of unnormalized 2D scale space derivatives (order 0-6). Each curve represents the ratio of variance of output to input noise of the linear unnormalized derivative of Gaussian filter vs. scale $\sigma$. Plot is on a log-log scale. 42

Figure 3.5. Plot of the propagated error of normalized 2D scale space derivatives (order 0-6). Each curve represents the ratio of variance of output to input noise of the linear normalized derivative of Gaussian filter vs. scale $\sigma$. 42

Figure 3.6. Plot of the propagated error of normalized 2D scale-space derivatives (order 0-6). Curve represents the ratio of variance of output to input noise of the linear unnormalized derivative of Gaussian filter vs. order of differentiation. Plot is on a log scale. ............................................................42

Figure 4.1a. Generic 1D square pulse function $P(d, x)$. Used as the input for generating pulse transfer functions. ..........................................................56

Figure 4.1b. 1D square pulse functions $P(1, x)$, $P(2, x)$, $P(4, x)$, $P(8, x)$. From left to right: $d = 1$, $d = 2$, $d = 4$, $d = 8$; $\lim_{d \to 0} P(d, x) = \delta(x)$. .........................................................57

Figure 4.2. 1D pulse transfer function for the multiscale mean operator $\mu_{P(d,x)}(x|\sigma)$. From left to right: $d = 1$, $d = 2$, $d = 4$, $d = 8$. In all images, $\sigma = 1$. The dashed lines represent the input pulse function $P(d,x)$. Note the difference in spatial and intensity ranges in each image. ..........................................................57

Figure 4.3. 1D pulse transfer function for the multiscale variance operator $\mu_{P(d,x)}^{(2)}(x|\sigma)$. From left to right: $d = 1$, $d = 2$, $d = 4$, $d = 8$. In all images, $\sigma = 1$. Note the difference in spatial and intensity ranges in each image. ..........................................................58
Figure 4.4. Comparison of the 1D pulse transfer function for the multiscale variance operator $\mu_{P(d,x)}^{(2)}(x|\sigma)$ to the square multiscale gradient magnitude operator. Top row, $\mu_{P(d,x)}^{(2)}(x|\sigma)$. Bottom row: $(\frac{\partial}{\partial x}P(d,x|\sigma))^2$. From left to right: $d = 1, d = 2, d = 4, d = 8$. In all images, $\sigma = 1$. Note the difference in spatial and intensity ranges in each image.59

Figure 4.5. 1D pulse transfer function of $\mu_{P(d,x)}^{(3)}(x|\sigma)$. From left to right: $d = 1, d = 2, d = 4, d = 8$. In all images, $\sigma = 1$. Note the difference in spatial and intensity ranges in each image.59

Figure 4.6. Comparison of $\mu_{P(d,x)}^{(3)}(x|\sigma)$ to $\frac{\partial}{\partial x}P(d,x|\sigma)$. Top row, $\mu_{P(d,x)}^{(3)}(x|\sigma)$. Bottom row: $(\frac{\partial}{\partial x}P(d,x|\sigma))^2$. From left to right: $d = 1, d = 2, d = 4, d = 8$. In all images, $\sigma = 1$. Note the difference in spatial and intensity ranges in each image.60

Figure 4.7. 1D pulse transfer function of $\mu_{P(d,x)}^{(4)}(x|\sigma)$. From left to right: $d = 1, d = 2, d = 4, d = 8$. In all images, $\sigma = 1$. Note the difference in spatial and intensity ranges in each image.61

Figure 4.8. Comparison of $\mu_{P(d,x)}^{(4)}(x|\sigma)$ to $(\frac{\partial}{\partial x}P(d,x|\sigma))^2$. Top row, $\mu_{P(d,x)}^{(4)}(x|\sigma)$. Bottom row: $(\frac{\partial}{\partial x}P(d,x|\sigma))^2$. From left to right: $d = 1, d = 2, d = 4, d = 8$. In all images, $\sigma = 1$. Note the difference in spatial and intensity ranges in each image.61

Figure 4.9. Comparisons of $\mu_{P(d,x)}^{(2)}(x|\sigma)$ with $\text{dog}(P(d,x); \sigma_a, \sigma_b)$. The input function is a pulse $P(d,x)$. In all cases, $d = 1$. From left to right: a. $\mu_{P(d,x)}^{(2)}(x|\sigma)$ with $\sigma = 1$, b. $\text{dog}(P(d,x); \sigma_a, \sigma_b)$ with $\sigma_a = \frac{\sigma_b}{\sqrt{2}}, \sigma_b = 1$, and c. $\text{Dog}(P(d,x); \sigma_a, \sigma_b)$ with $\sigma_a = 0, \sigma_b = 1$.64

Figure 4.10. Test function $T(h,x)$.67

Figure 4.11. A 128 x 128 pixel Teardrop with Signal to Noise of 4:1.69

Figure 4.12. Local statistical measure of the teardrop from Fig. 4.11.69

Figure 4.13. A test object. The figure contains structures at different scale. The raster resolution of the object is 128 x 128 pixels.72

Figure 4.14. Results from the modified multiscale statistical approach to vcd (left: initial image, right: after 75 iterations of vcd).72

Figure 4.15. Early work in statistically driven multivalued vcd. A synthetic multivalued image where the values are subject to significant gaussian white noise and with a strong negative correlation between intensity values. (a) - original two valued input image and its scatterplot histogram. (b) - image after processing with vcd and resulting histogram.76

Figure 5.1. - A test image with SNR of 4:1 with a raster resolution of 256 x 256 pixels.87

Figure 5.2. Directional variances of the objects from Figure 5.1. (From left to right: a: $V_{xx} = \mu_{1,xx}^{(2)}(p|\sigma)$, b: $V_{xy} = \mu_{1,xy}^{(2)}(p|\sigma)$, c: $V_{yy} = \mu_{1,yy}^{(2)}(p|\sigma)$). In all images, a grey value is 0, and $\sigma = 2$ pixels. Bright grey to white indicates positive values, and dark grey to black indicates negative values. Each image uses a left handed coordinate system with the origin in the upper left corner, the x-axis oriented to the right, and the y-axis oriented from the top.88
Figure 5.3. Eigenvalue images of the object from Figure 5.1, computed with a spatial aperture or scale $\sigma$ of 2 pixels. (From left to right: a: $\lambda_1$, b: $\lambda_2$). In both of these images, black is zero and bright indicates positive values.........................................91

Figure 5.4. Eigenvector image of the object from Figure 5.1, computed with a spatial aperture or scale $\sigma$ of 2 pixels. The image reflects only the eigenvector $u$ in the direction of maximum variance at each pixel; the eigenvector $v$ in the direction of minimum variance is perpendicular to the vectors shown. The lengths of the vector representations indicate relative magnitude..............................................................92

Figure 5.5. (a) Test figure exhibiting significant directional spatial correlation and (b) the local anisotropy statistic $\hat{Q}$ where $\sigma = 3$. In both images, the raster resolution is $256 \times 256$. .........................................................................................................................95

Figure 6.1. A 2D dual-echo MR image of the head with its scatterplot histogram. ......112
Figure 6.2. MR image of a shoulder acquired using a surface receiving coil. This may represent the ultimate test for this research.................................................................114
**List of Symbols**

Symbols are listed in order of their appearance in the text.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{R}$</td>
<td>real numbers</td>
</tr>
<tr>
<td>$\mathbb{R}^n$</td>
<td>n-space of real numbers</td>
</tr>
<tr>
<td>$\rightarrow$</td>
<td>maps onto</td>
</tr>
<tr>
<td>$\mathbf{p}$</td>
<td>bold designates vector or tensor quantities</td>
</tr>
<tr>
<td>$I(\mathbf{p})$</td>
<td>function $I$ of $\mathbf{p}$</td>
</tr>
<tr>
<td>$\in$</td>
<td>is an element of</td>
</tr>
<tr>
<td>$\mathbb{N}$</td>
<td>natural numbers, (i.e., 0, 1, 2, 3, ...</td>
</tr>
<tr>
<td>$\mathbb{N}^n$</td>
<td>n-space of natural numbers</td>
</tr>
<tr>
<td>$\subset$</td>
<td>is a subset of</td>
</tr>
<tr>
<td>$L$</td>
<td>italics designate set notation</td>
</tr>
<tr>
<td>$\mathbf{V}$</td>
<td>a vector field $\mathbf{V}$</td>
</tr>
<tr>
<td>$\cdot$</td>
<td>dot or inner product operator</td>
</tr>
<tr>
<td>$\nabla I$</td>
<td>gradient of $I$</td>
</tr>
<tr>
<td>$\theta$</td>
<td>Greek lower-case theta, used to designate an angular value</td>
</tr>
<tr>
<td>$\otimes$</td>
<td>the convolution operator</td>
</tr>
<tr>
<td>$G(\sigma, \mathbf{p})$</td>
<td>Gaussian with spatial scale $\sigma$.</td>
</tr>
<tr>
<td>$\int$</td>
<td>integral</td>
</tr>
<tr>
<td>$\frac{\partial^n}{\partial x^n}$</td>
<td>n-th partial derivative with respect to $x$</td>
</tr>
<tr>
<td>$\Sigma$</td>
<td>Greek upper-case sigma, without subscripts, denotes a covariance matrix</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>Greek lower-case sigma, used as the scale parameter</td>
</tr>
<tr>
<td>$e$</td>
<td>the transcendental value $e$, the natural logarithm</td>
</tr>
<tr>
<td>$\pi$</td>
<td>Greek lower-case pi, the transcendental value, $\pi$</td>
</tr>
<tr>
<td>$\mathbf{p}^T$</td>
<td>transpose of the tensor value $\mathbf{p}$</td>
</tr>
<tr>
<td>$\nabla \cdot \mathbf{F}$</td>
<td>“del-dot,” the divergence of $\mathbf{F}$</td>
</tr>
<tr>
<td>$I(\mathbf{p}</td>
<td>\sigma)$</td>
</tr>
<tr>
<td>$\tilde{u}$</td>
<td>tilde over a character denotes random variable</td>
</tr>
<tr>
<td>$\mu$</td>
<td>Greek lower-case mu, used to designate moments</td>
</tr>
<tr>
<td>$\langle \rangle$</td>
<td>the expectation operation</td>
</tr>
<tr>
<td>$N_{\mu, \sigma^2}(\tilde{u})$</td>
<td>Standard Normal distribution of $\tilde{u}$, with mean $\mu$ and variance $\mu^{(2)}$.</td>
</tr>
<tr>
<td>$\mu^{(n)}$</td>
<td>n-th central moment</td>
</tr>
<tr>
<td>$\sum_{j=1}^{n} f(j)$</td>
<td>summation of $f(0) + f(1) + ... + f(n)$</td>
</tr>
<tr>
<td>$i$</td>
<td>imaginary value $\sqrt{-1}$</td>
</tr>
<tr>
<td>$\overline{M}(u,v)$</td>
<td>the u-v spatial moment of an image function</td>
</tr>
<tr>
<td>$L_{x'y''}^{n+m}$</td>
<td>multiscale partial derivative $\frac{\partial^n}{\partial x^n} \frac{\partial^m}{\partial y^m} G(\sigma, \mathbf{p}) \otimes I(\mathbf{p})$</td>
</tr>
<tr>
<td>$\hat{L}_{x'y''}^{n+m}$</td>
<td>normalized multiscale partial derivative $\sigma^{n+m} \frac{\partial^n}{\partial x^n} \frac{\partial^m}{\partial y^m} G(\sigma, \mathbf{p}) \otimes I(\mathbf{p})$</td>
</tr>
<tr>
<td>$\mathbb{Z}$</td>
<td>integers</td>
</tr>
</tbody>
</table>
\( \forall \) for all
\( V(\tilde{u}) \) variance of \( \tilde{u} \)
\( M(\tilde{u}) \) mean of \( \tilde{u} \)
\( \text{Cov}(\tilde{u}) \) covariance of \( \tilde{u} \)
\( \prod_{k=1}^{n} f(k) \) product of \( f(0) + f(1) + \ldots + f(n) \)
\( F \) stochastic process \( F \)
\( f(x) \xrightarrow{a \to \infty} c \) \( f(x) \) approaches \( c \) as \( a \) goes to infinity
\( \lim_{d \to 0} P(d) \) limit of \( P(d) \) as \( d \) approaches 0
\( \text{erf}(x) \) standard error function: \( \text{erf}(x) = \int_{-\infty}^{x} G(1, \tau) \, d\tau \)
\( |p| \) norm of \( p: p \cdot p \)
\( \perp \) perpendicular to
\( \max_{-\infty < x < \infty} (S(x)) \) global maximum of \( S(x) \) over the interval \( -\infty < x < \infty \)